On the Diophantine Equation $x^{n+1} + y^{n+1} = z^{n+1} + k^{m+1}$

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ABSTRACT

On the Diophantine equation

$x^{n+1} + y^{n+1} = z^{n+1} + k^{m+1}$, we analyse the integral solution for some value of m and n with k = 2. Observation found were recorded and presented.

KEYWORDS

Diophantine equation, integral solutions.

INTRODUCTION

The theory of Diophantine equations plays a significant role in higher arithmetic and has a marvelous effect on credulous people and always occupies a remarkable position due to unquestioned historical importance. Mathematicians worldwide have made a lot of discoveries, provided a huge number of theorems and have also deduced many amazing results[1 – 5].

In [6 – 9], the Diophantine equations for which we require integral solutions or algebraic equations with integer coefficient. In this context, one may refer [10 – 14] for various choices of the ternary quadratic Diophantine equations.

In this communication, we discussed with problem of obtaining infinitely many non-trivial integral solutions on the diophantine equation $x^{n+1} + y^{n+1} = z^{n+1} + k^{m+1}$, for particular values of k, m and n. A few results were presented.

METHOD

On the Diophantine equation

$x^{n+1} + y^{n+1} = z^{n+1} + k^{m+1}$, we discuss the integral solutions for particular values of k, m and n.

In this paper, we take k=2 and proceed with the equation. In choice 1, we consider m = 1 and n = 1, two different patterns are presented to find the integral solutions of equation I and various results are illustrated. Some numerical examples are also discussed, In choice 2, taking m = 2 and n = 1, another two patterns are presented, with numerical examples.

CHOICE 1:

Equation is reduced to ternary quadratic diophantine equation, To find the integral solutions, the pattern are presented below.

PATTERN 1:

Assuming the value z = Rx in the ternary quadratic diophantine equation, we get a equation

$y^2 = (R^2 - 1)x^2 + 1$ by considering the pellian $y_0 = R$, $x_0 = 1$.

The general solution (x_s, y_s) is given by

$$y_s + \sqrt{(R^2 - 1)}x_s = 2[R + \sqrt{(R^2 - 1)}]^{s+1}$$

$$y_s - \sqrt{(R^2 - 1)}x_s = 2[R - \sqrt{(R^2 - 1)}]^{s+1}, \quad s = 0, 1, 2, \ldots$$

because irrational roots occur in pairs,

Solving the above two equations, we get

$$Y_s = [\left(R + \sqrt{(R^2 - 1)}\right)^{s+1} + (R - \sqrt{(R^2 - 1)})^{s+1}] \quad \rightarrow \text{I}$$

$$x_s = \frac{1}{\sqrt{(R^2 - 1)}}[\left(R + \sqrt{(R^2 - 1)}\right)^{s+1} - (R - \sqrt{(R^2 - 1)})^{s+1}] \quad \rightarrow \text{II}$$

Substituting in the assumption

$z_s = \frac{R}{\sqrt{(R^2-1)}}$

$[(R + \sqrt{(R^2-1)})^{s+1} - (R - \sqrt{(R^2-1)})^{s+1}]$ → III

ILLUSTRATION:

- $y_{4s+3} = y_s^4 - 4y_s^2 - 16$
- $y_{2s+1} - y_s^2 + 2 = 0$
- $y_{3s+2} = y_s^3 - 3y_s$
- $z_{3s+2} = z_s[y_s^2 - 1]$
- $2z_{2s+1=(R+1)}y_s[x_s + z_s]$
- $\frac{2z_{2s+1}}{x_{2s+1}} = (R+1)^2$
- $x_{2s+1} = y_s x_s$
- $(R+1)x_{2s+1} = y_s[x_s + z_s]$

Recurrence relation satisfied by the solution are found by

$y_{s+2} - 2Ry_{s+1} + y_s = o$, $x_{s+2} - 2Rx_{s+1} + x_s = o$ and $z_{s+2} - 2Rz_{s+1} + z_s = o$

PATTERN 2:

Assuming the value of x as y + k, the general equation leads to

$(2y+k)^2 = 2z^2 + 8 - k^2$ → IV

In order to reduce the equation into the well known pellian equation taking X = 2y +k. That is, to obtain the other solutions whose least positive integer solutions of IV is $y_0 = 2$, $x_0 = k + 2$ and $z_0 = k + 2$.

Pellian equation is $X^2 = 2z^2 + 1$ whose general solution ($\tilde{X}_s, \tilde{z}_s$) can be simplified as the solution

$\tilde{X}_s + \sqrt{2}\,\tilde{z}_s = (3+2\sqrt{2})^{s+1}$, s = 0,1,2,……

$\tilde{X}_s - \sqrt{2}\,\tilde{z}_s = (3-2\sqrt{2})^{s+1}$, s = 0,1,2,…….

By solving the above two equation

$\tilde{X}_s = \frac{1}{2}[(3+2\sqrt{2})^{s+1} + (3-2\sqrt{2})^{s+1}]$

and hence

$\tilde{z}_s = \frac{1}{2\sqrt{2}}[(3+2\sqrt{2})^{s+1} - (3-2\sqrt{2})^{s+1}]$

Applying Brahmagupta lemma between the solutions (X_0, z_0) and ($\tilde{X}_s, \tilde{z}_s$) the other solutions are presented as

$X_{s+1} = \frac{1}{2\sqrt{2}}[\sqrt{2}(k+4)U + 2(k+2)V]$

$z_{s+1} = \frac{1}{2\sqrt{2}}[\sqrt{2}(k+4)U + (k+4)V]$ → V

where $U = (3+2\sqrt{2})^{s+1} + (3-2\sqrt{2})^{s+1}$ and

$V = (3+2\sqrt{2})^{s+1} - (3-2\sqrt{2})^{s+1}$

y_{s+1} were simplified from the results which are already available.Therefore,

$y_{s+1} = \frac{1}{2}[\frac{1}{2\sqrt{2}}\{\sqrt{2}(k+4)U + 2(k+2)V\} - k]$ → VI

Since we are interested in finding the integral solution of x_{s+1}, by substituting y_{s+1}, in the assumption we get

$x_{s+1} = \frac{1}{2}[\frac{1}{2\sqrt{2}}\{\sqrt{2}(k+4)U + 2(k+2)V\} - k] + k$

→ VII

Therefore, the integral solution for the diophantine equation considered are V, VI and VII.

NUMERICAL EXAMPLE

Numerical examples are given below when k takes even values k = 2, 4 and 6.

k = 2

	x_{s+1}	y_{s+1}	z_{s+1}
S=0	18	16	24
S=1	100	98	140
S=2	578	576	816

S=3	3364	3362	4756
S=4	19602	19600	27720
S=5	114244	114242	161564

k = 4

	x_{s+1}	y_{s+1}	z_{s+1}
S=0	26	22	34
S=1	142	138	198
S=2	818	814	1154
S=3	4758	4754	6726
S=4	27722	27718	39202
S=5	161566	161562	228486

k = 6

	x_{s+1}	y_{s+1}	z_{s+1}
S=0	34	28	44
S=1	184	178	256
S=2	1058	1052	1492
S=3	6152	6146	8696
S=4	35842	35836	50684
S=5	208888	208882	295408

RESULTS:

- $\sum(x_{s+1} + y_{s+1} + z_{s+1}) = \sum x_{s+1} + \sum y_{s+1} + \sum z_{s+1}$ where s=0, 1, 2, ...
- $\sum(x_{s+1} + y_{s+1} + z_{s+1})$,

  $\sum x_{s+1}, \sum y_{s+1}, \sum z_{s+1} \equiv 0 \pmod 2$ where

  s=0, 1, 2, ...
- The difference between the summation of $x_{s+1}, y_{s+1}, z_{s+1}$ for k = 6 and for k = 4 is same as for k = 4 and k = 2.

CHOICE 2:

Two different patterns are exhibited to find the integral solution of ternary diophantine equation.

PATTERN 1:

Assuming the value of x as y + 3, the ternary diophantine equation is reduced to

$$(2y + 3)^2 = 2z^2 + 7$$

Applying the similar method followed in the previous section, we get

$$y_s = \frac{1}{4}[3R + \sqrt{2}\,S - 6]$$

$$z_s = \frac{1}{2\sqrt{2}}[\sqrt{2}\,R + 3S]$$

and hence $x_s = \frac{1}{4}[3R + \sqrt{2}\,S + 6]$

R and S are taken as

$$R = (3 + 2\sqrt{2})^{s+1} + (3 - 2\sqrt{2})^{s+1},$$

$S = (3 + 2\sqrt{2})^{s+1} - (3 - 2\sqrt{2})^{s+1}$ Therefore the non-zero integral solutions of ternary diophantine equations are

$$x_s = \frac{1}{4}[3R + \sqrt{2}\,S + 6]$$

$$y_s = \frac{1}{4}[3R + \sqrt{2}\,S - 6]$$

$$z_s = \frac{1}{2\sqrt{2}}[\sqrt{2}\,R + 3S]$$

The recurrence relation satisfied by the values x_s, y_s and z_s are expressed as

$x_{s+2} - 6x_{s+1} + x_s = -6$, $y_{s+2} - 6y_{s+1} + y_s = 6$ and $z_{s+2} - 6z_{s+1} + z_s = 0$.

The initial values are

$x_0 = 8, x_1 = 39, y_0 = 5, y_1 = 36, z_0 = 9, z_1 = 53$.

Numerical values recorded are

s	x_s	y_s	z_s
0	8	5	9
1	39	36	53
2	220	217	309
3	1275	1272	1801
4	7224	7421	10497

5	43263	43260	61181

Analysing these numerical values, the results found among them are presented here.

- $x_{2s} \equiv 0 \pmod 4)$ s= 0, 1, 2, 3, ...
- $z_s + (2\beta + 1) \equiv 0 (\text{mod } 2)\ \beta$, s = 0, 1, 2,3,..........
- $7(12x_{2s+1} - 4z_{2s+1} - 4) = (12x_s - 4z_s - 18)^2$
- $(2x_s - 3)^2 - 2{z_s}^2 = 7$

It is observed that $W = 6x_s - 2z_s - 9$,

$T = 2x_s - 3z_s - 3$ satisfy the diophantine equation $W^2 = 2X^2 + 7^2$.

PATTERN 2:

In order to find the non-zero integral solutions of ternary diophantine equation, the linear transformation is applied as z = r + s, x = r – s, by which the equation will reduce to

$$y^2 = 4rs + 8.$$

r ans s are distinct non-zero parameters. It is possible to choose r and s such that rs+2 is a square and the value of y is obtained.

The solutions of x, y and z were found.

Numerical illustration are elaborated in the following table as given below.

r	s	x	y	z
2	7	-5	8	9
$n^2 + 2n - 1$	1	$n^2 + 2n - 2$	2n+2	$n^2 + 2n$
2	$2n^2 - 1$	$3 - 2n^2$	4n	$2n^2 + 1$
$2n^2 + 4n + 1$	2	$2n^2 + 4n - 1$	$4n + 4$	$2n^2 + 4n + 3$

CONCLUSION:

In this communication, we tried to find the non-zero integral solutions of ternary diophantine equation for particular values of k, m and n. Few patterns and some relations are observed. Numerical examples are also discussed, In extension of this, one may search for other patterns.

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