Laplace transforms and it's Applications in Engineering Field

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Abstract

Laplace transform is a powerful mathematical technique useful to the engineers and scientists, as it enables them to solve linear differential equations with given initial conditions by using algebraic methods. The concept of laplace transform are applied in area of science and technology such as electric analysis communication engineering, control engineering, linear system analysis statistics optics , quantum physics etc. In solving problems relating to there fields, one usually encounters problems on time invariant, differential equations, time frequency domain for non periodic wave forms. This paper provides the reader to know about the fundamentals of laplace transform and gain an understanding of some of the very important and basic applications of there fundamental to engineering field.

Keywords : Laplace functions , Dirac delta functions , Inverse laplace , linearity.

I. Introduction

The solution of physical problem has been a challenge to the scientist and engineers alike. The analysis of engineering field and solutions of linear differential equations is simplified by use of laplace transform .The laplace transform provides a method of analysing a linear system using algebraic methods. The basic process of analysing a system using laplace transform involves conversion of the system transfer function or differential equation into s-domain , using s-domain to convert input functions, finding an output function by algebraically combing input and transfer functions , using partial functions to reduce the output function to simpler components and conversion of output equation back to time domain. In order for any function of time t ie., f (t) to be laplace transformable it must satisfy the following Dirichlet conditions.

* f(t) must be piecewise continous which means that it must be single valued but can have a finite number of finite isolated discontiniuities for t > 0

* f(t) must be exponential order which means that f(t) must remain less than $Se^{-a_0 t}$ as t approaches ∞ where s is a positive constant and a_o is a real positive number.

If there is any function f(t) that satisfies the Dirichlet conditions , then

$$F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

is called the laplace transformation of f (t). Here s can be either a real variable or a complex quantity.

The integral $\int f(t) e^{-st}\, dt$ converges if $\int | f(t) e^{-st} | dt \prec \infty$

A:- Some important properties of laplace transforms

Given the functions f (t) and g (t) and their respective laplace transforms F (s) and G (s)

$f(t) = L^{-1}[F(s)]$, $g(t) = L^{-1}[G(s)]$

The following is a list of properties of unilateral laplace transform.

- Linearity
  $L[a f(t) + b g(t)] = a F(s) + b G(s)$
- Frequency differentiation

  $L[t f(t)] = -F'(s)$

  $L[t^n f(t)] = (-1)^n F^n(s)$

- Differentiation '
  $L[f'] = s L[f] - f(0^-)$
  $L[f''] = s^2 L[f] - s f(0^-) - f'(0^-)$
  $L[f^{(n)}] = s^n L[f] - s^{n-1} f(0^-) - \ldots\ldots - f^{n-1}(0^-)$
- Frequency integration
  $$L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(\sigma)\, d\sigma$$
- Frequency shifting

$$L[e^{at} f(t)] = F(s-a)$$
$$L^{-1}[F(s-a)] = e^{at} f(t)$$

- Time shifting

$$L[f(t-a)u(t-a)] = e^{-as}F(s)$$
$$L^{-1}[e^{-as}F(s)] = f(t-a)u(t-a)$$

where u (t) is the Heaviside step function.

B. Relationship to other transforms

Fourier transform

The continous Fourier transform is equivalent to evaluating the bilateral laplace transform with complex argument s = iw

$$F(\boldsymbol{\omega}) = F[f(t)] = L[f(t)]/_{s=iw}$$
$$= F(s)/_{s=iw}$$
$$= \int_{-\infty}^{\infty} e^{-iwt} f(t)dt$$

Note that this expression excludes the scaling factor $\frac{1}{\sqrt{\pi}}$ which is often included in definition of the fourier transform.

This relationship between the laplace and fourier transform is often used to determine the frequency spectrum of a signal or dynamical system.

Mellin transform

The Mellin transform and its inverse are related to the two-sided laplace transform by a simple change of variables. If in the Mellin transform

$$G(s) = M(g(\boldsymbol{\theta})) = \int_{\mathbf{0}}^{\infty} \boldsymbol{\theta}^{s} \boldsymbol{g}(\boldsymbol{\theta}) \frac{\boldsymbol{d\theta}}{\boldsymbol{\theta}}$$

We set $\boldsymbol{\theta}$ = exp (- t) we get a two sided laplace transform.

Z transform

The z transform is simply the laplace transform of an ideally sampled signal with substitution of $Z \equiv e^{sT}$ where $T = 1/f_s$ is the sampling period (in units of time e.g. seconds) and f_s is the sampling rate(in samples per second or hertz)

Let $\Delta_T(t) \equiv \sum_{n=0}^{\infty} \delta(t-nT)$ be a sampling impulse train (also called a Dirac comb) and

$$x_q(t) \equiv x(t)\Delta_T(t) = x(t) \sum_{n=0}^{\infty} \delta(t-nT)$$

$$= \sum_{n=0}^{\infty} x(nT)\, \delta(t-nT) = \quad = \sum_{n=0}^{\infty} x[n]\, \delta(t-nT)$$

be the continous-time representation of the sampled x (t).

x [n] = x (n T) are the discrete samples of x(t) . The laplace transform of the sampled signal $x_q(t)$ is

$$X_q(s) = \int_{0-}^{\infty} x_q(t)e^{-st}\, dt$$

$$= \int_{0-}^{\infty} \sum_{n=0}^{\infty} x[n]\, \delta(t-nT)e^{-st}dt$$

$$= \sum_{n=o}^{\infty} x[n] \int_{o-}^{\infty} \delta(t-nT)e^{-st}\, dt$$

$$= \sum_{n=0}^{\infty} x[n]\, e^{-nst}$$

This is precisely the definition of the z-transform of the discrete function x [n].

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

with the substitution of $z \leftarrow e^{sT}$

Comparing the last two equations , we find the relationship between the z-transform and the laplace transform of the sampled signal.

$$X_q(s) = X(z)/_{z=e^{sT}}$$

Fundamental relationships

Since an ordinary laplace transform can be written as a special case of a two-sided transform and since the two-sided transform can be written as the sum of two one-sided transforms the theory of the laplace , fourier , mellin and z-transform are at bottom of the same subject. However a different point of view and different characteristic problems are associated with each of these four major integral transforms.

II.Applications of laplace transform

This section describes the applications of laplace transform in the domain of engineering. Laplace transform has several applications in almost all Engineering disciplines such as System Modelling, Analysis of Electrical and Electronic Circuits, Digital

Signal Processing and Process Controls. Laplace transforms are critical for process controls. It helps analyze the variables, which when altered, produces desired manipulations in the result. For example, while study heat experiments, Laplace transform is used to find out to what extent the given input can be altered by changing temperature, hence one can alter temperature to get desired output for a while. This is an efficient and easier way to control processes that are guided by differential equations

Electric Circuit theory

Example 1

The switching transient phenomenon in the RL , RC or RLC circuits can be solved by laplace transform. Let us consider a series RLC circuit as shown figure 1 to which a dc voltage V_0 is suddenly applied.

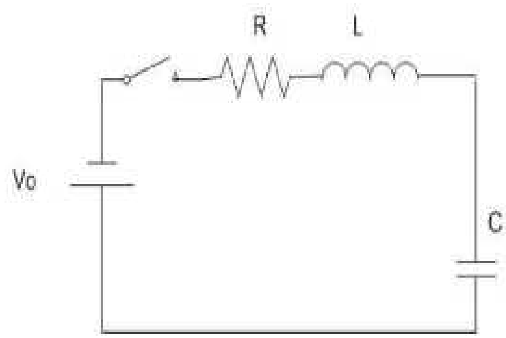

FIG. 1: Series RLC circuit

Now applying kirchoff's voltage law (KVL) to the circuit , we have

$$R_i + L\frac{d\,i}{d\,t} + \frac{1}{C}\int i\,dt = V_0$$

Differentiating bothsides,

$$L\frac{d^2 i}{dt^2} + \frac{1}{C\,i} + R\frac{d\,i}{d\,t} = 0 \;\; or$$

$$\frac{d^2 i}{d\,t^2} + \frac{R}{L}\frac{d\,i}{d\,t} + \frac{1}{L\,C}\,i = 0 \quad \rightarrow (1)$$

Applying laplace transform to this equation, let us assume that the solution of this equation is

$$i(t) = K\,e^{sT}$$

where K and s are constants which may be real, imaginary or complex.

From equation (1) ,

$$L\,K\,s^2\,e^{s\,T} + R\,K\,e^{s\,T} + \frac{1}{C\,K}\,e^{s\,T} = 0$$

which on simplification gives

$$s^2 + \frac{R}{L}\,s + \frac{1}{L\,C} = 0$$

The two roots of this eqution would be

$$s_1, s_2 = \frac{R}{2\,L} \pm \sqrt{\frac{R^2}{4\,L^2} - \frac{1}{L\,C}}$$

The general solution of the differential equation is thus ,

$$i(t) = K_1\,e^{s_1 T} + K_2\,e^{s_2 T}$$

where K_1 and K_2 are determined from the initial conditions.

Now if we define

α = Damping coefficient = R / 2 L

and Natural frequency $w_n = \frac{1}{\sqrt{L\,C}}$

which is also known as undamped natural frequency or resonant frequency.

Thus the roots are $s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - {w_n}^2}$

The final form of solution depends on whether

$$\left(\frac{R^2}{4L^2}\right) > \frac{1}{L\,C} \quad \left(R = \frac{1}{L\,C}\right)$$

Example 2

In the theory of electrical circuits, the current flow in a capacitor is proportional to the capacitance and the rate of change in the electrical potential in SI units.

Symbollically, this is expressed by the differential equation

$$i = C\,\frac{d\,v}{d\,t}$$

where C is the capacitance (in farads) of the capacitor, i = i (t) is the electric current (in ampheres) through the capacitor as a function of time and v = v (t) is the voltage(in volts) across the terminals of the capacitor, also a function of time.

Taking Lapalace transform,

$$I(s) = C[sV(s) - V_0]$$

where $I(s) = L[i(t)]$,

$V(s) = L[v(t)]$, $V_0 = V(t)/_{t=0}$

$$V(s) = \frac{I(s)}{sC} + \frac{V_0}{s} \rightarrow (2)$$

The definition of the complex impedance Z (in ohms) is the ratio of the complex voltage V divided by the complex current I while holding the initial state V_0 at zero:

$$Z(s) = \frac{V(s)}{I(s)} /_{V_0=0}$$

Using this definition and equation (2) , $Z(s) = \frac{1}{sC}$

which is the correct expression for the complex impedance of a capacitor.

Conclusion

Relations of laplace transform with other transforms are discussed in this paper and it presented the application of Laplace transform in engineering field. Besides these, Laplace transform is a very effective mathematical tool to simplify very complex problems in the area of stability and control.With the ease of application of Laplace transforms in myriad of scientific applications, many research software's have made it possible to simulate the Laplace transformable equations directly. This has made a good advancement in the research field.

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